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GRADE 11

The following math problems are representative of what 11^{th}-grade students in India are expected to master to be promoted to grade 12.

As an 11^{th}-grader here in America, we hope you will try out some or all of these problems. If you are stuck or are not satisfied with your solutions, you can always ask for help from your peers and friends through BeyondGPA's q&a service.

Note the introduction of set theory in the 11^{th}-grade math curriculum in India. If it is new and strange to you, don't worry. With a little bit of patience and practice, you can master the basic concepts and solve rudimentary set theory problems yourself. Besides, a student set theory expert is waiting somewhere in America to hear from you.

You should also monitor questions posted by your peers and help if you can. After all, it is students who can best help other students.

Shown below are some symbols used in set theory. Also shown are few common trigonometric identities to help you with solving the problems. Look up more identities and formulas as you need them.

$\in$ = member of
(If 'a' is a member of Set A, then a $\in$ A, which means 'a' belongs to A.)
$\notin$ = not a member of
$\subseteq$ = subset $\subset$ = proper subset
$\supseteq$ = superset $\supset$ = proper superset
$\not\subset$ = not subset
Ø = null or empty set. It has no elements.

$\cup$ = union
The union of A and B is represented by A $\cup$ B which is the set of all elements that belong to either A or B or both A and B. Thus A $\cup$ B (read as "A union B") = {x| x $\in$ A or x $\in$ B}

$\cap$ = intersection
The intersection of A and B is represented by A $\cap$ B which is the set of all elements that belong to both A and B. Thus A $\cap$ B (read as "A intersection B") = {x: x $\in$ A and x $\in$ B}

Two sets A and B are disjoint if A $\cap$ B = Ø

The following are subsets of the set R of real numbers:
The set of all natural numbers N = {1,2,3,4,5,6,7, …}
The set of all integers Z = {… -3, -2, -1, 0, 1, 2, 3, …}

The set of all rational numbers Q = $\{x \mid x = m/n,\ m, n \in Z,\ n \neq 0\}$
The set of all irrational numbers T = $\{x \mid x \in R \text{ and } x \notin Q\}$

Expressing a set in roster form:
{ } where set elements within braces are separated by commas.
Example: A = {2, 3, 5, 7, 11, 13} represents a set of prime numbers less than 17
Example: Set of odd natural numbers can be described as {1, 3, 5, 7, …} where the ellipses stand for 'and so on.'

Expressing a set in set-builder form:
{x: P(x) holds} which reads: 'the set of all x such that P(x) holds'
The set of numbers 1 to 12 can be written as A = $\{x \in N \mid x <= 12\}$ where N represents the set of natural numbers.
The set A = {0, 1, 4, 9, 16, 25, …} can be written as A = $\{x^2 \mid x \in Z\}$ where Z represents the set of integers.

Few trigonometric identities:

$\sin^2\theta + \cos^2\theta = 1$	$1 + \cot^2\theta = \mathrm{cosec}^2\theta$
$\tan\theta = \sin\theta/\cos\theta$	$(\tan\theta)(\cot\theta) = 1$
$1 + \tan^2\theta = \sec^2\theta$	$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(-\theta) = -\sin\theta$	$\cos(A+B) = \cos A\cos B - \sin A\sin B$

1. Prove the following identities:

   a) $(\tan\theta + \sec\theta - 1)/(\tan\theta - \sec\theta + 1) = (1 + \sin\theta)/\cos\theta$
   b) $\cot^4\theta + \cot^2\theta = \mathrm{cosec}^4\theta - \mathrm{cosec}^2\theta$
   c) $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$
   d) $\tan 3A = (3\tan A - \tan^3 A)/(1 - 3\tan^2 A)$
   e) $\cos 3A = 4\cos^3 A - 3\cos A$
   f) $\cos 2\theta/(1 + \sin 2\theta) = \tan(\pi/4 - \theta)$

2. Find the values of the following trigonometric ratios:

   a) $\cos(-480^0)$
   b) $\sin 315^0$
   c) $\tan 480^0$
   d) $\mathrm{cosec}\ 390^0$

3. Prove that

   a) $\tan(45^0 + x)/(\tan(45^0 - x) = \{(1 + \tan x)/(1 - \tan x)\}^2$

   b) If $\cos A = 4/5$ and $\cos B = 12/13$, $3\pi/2 < A, B < 2\pi$, find the value of
   i) $\sin(A - B)$ ii) $\cos(A + B)$

4. Solve the following trigonometric equations:

   a) $2\sin^2 x + \sin^2 2x = 2$

   b) $7\cos^2 x + 3\sin^2 x = 4$

c) $\cot \theta + \text{cosec } \theta = \sqrt{3}$ (square root of 3)

5. In a triangle ABC, a represents the side BC, b represents the side CA and c represents the side AB. A represents the angle BAC, B represents the angle ABC and C represents the angle BCA. Prove the following:

   a) a/sin A = b/sin B = c/sin C

   b) sin (B-C)/sin (B+C) = $(b^2 - c^2)/a^2$

   c) a sin (B-C)/(b^2-c^2) = b sin (C-A)/(c^2-a^2) = c sin (A-B)/(a^2-b^2)

6. The square root of -1 ($\sqrt{-1}$) is an imaginary number represented by the symbol i. You can think of it as the solution of $x^2 + 1 = 0$ where $i^2 = -1$. It was introduced by the great Swiss mathematician Euler (pronounced "Oiler", 1707-83). If a and b are real numbers, a + bi is called a complex number.

   a) Express (5 + 4i)/(4 + 5i) in the form a + bi

   b) Find the inverse of $(2 + \sqrt{-3})^2$ and express your result in the form a + bi

   c) If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$

   d) One of the most famous and compact formulas in mathematics is known as Euler's identity: $e^{i\pi} + 1 = 0$. Prove Euler's identity. The identity is important because it connects complex numbers to trigonometry. It also links five of the most important numbers in mathematic - 0, 1, i, e and π - in which the last three symbols are due to Euler. You can also see that the identitiy correlates three key mathematical operations: addition, multiplication and exponentiation. Look up other variations of the identity on the Internet, some of which are attributed to French mathematician De Moivre (1667-1754).

7. Solve the following quadratic equations. You will need to use the concept of i.

   a) $x^2 - 4x + 13 = 0$

   b) $25x^2 - 30x + 11 = 0$

   c) $3x^2 + 7ix + 6 = 0$

8. Solve the following linear inequalities and graph the solutions on a line.

   a) (x+3)/4 >= (x-2)/3 + 1/4

   b) (2x + 4)/(x – 1) >= 5

9. a) Write the set $A = \{x | x \in Z, x^2 < 40\}$ in the roster form, where Z represents the set of integers.

   b) Write the set A = {1, 1/4, 1/9, 1/16, 1/25, …} in the set builder form.

   c) Write the set of all vowels in the English alphabet which precede s.

d) Write the set of all positive integers whose cube is odd.

e) Write the set {1/2, 2/5, 3/10, 4/17, 5/26, 6/37, 7/50, 8/65} in set builder form.

f) Write the set $A = \{a_n \mid n \in N, a_{n+1} = 3a_n \text{ and } a_1 = 2\}$ in roster form.

g) Write the set $A = \{a_n \mid n \in N, a_{n+2} = a_{n+1} + a_n, a_1 = a_2 = 1\}$ in roster form. What famous mathematical series does this represent?

10. a) If A = {1,2,3,4,5,6,7,8} and B = {2,4,8,16}, find $A \cap B$.

    b) If A = {1,2,3,4,5}, B = {4,5,6,7}, C = {7,8,9,10,11} and D = {10,11,12,13,14}, find

    i) $A \cup B$
    ii) $A \cup B \cup C$
    iii) $B \cup C \cup D$
    iv) $A \cap (B \cup C)$
    v) $(A \cup D) \cap (B \cup C)$

11. In a group of 950 Indians, 750 can speak Hindi and 460 can speak English.

    a) How many people can speak both Hindi and English?
    b) How many people can speak Hindi only?
    c) How many people can speak English only?

12. Set theory, combined with probability theory, is a fertile field of research. Use the Internet to explore the applications of set theory to different branches of mathematics and physics. Find out about Venn Diagrams, Relations, Functions and Domains. Look up the contributions of German mathematician Georg Cantor (1845-1918) and English mathematicians George Boole (1815-1864) and Bertrand Russell (1872-1970) to set theory.

13. n! ("n factorial") is defined as the product of the first n natural numbers. Thus n! = 1 x 2 x 3 x 4 x 5 … x (n-1) x n. Find

    a) 15!/(6! 3!)

    b) (18! 12!)/(8! 11!)

    c) Find n if (n+1)! = 12 x (n-1)!

    d) Prove that n! (n+2)! = n! + (n+1)!

14. a) In a midterm math test, the teacher decides that there will be a total of 6 questions, one each from chapters 5, 6, 7, 8, 9, 10. There are 8 questions in chapter 5, 12 in chapter 6, 10 in chapter 7, 14 in chapter 8, 6 in chapter 9 and 11 in chapter 10. In how many ways can the teacher select 6 questions?

    b) How many numbers are there between 100 and 1000 in which all the digits are distinct?

15. If n and r are positive integers such that $1 <= r <= n$, the number of all permutations of n distinct objects, taken r at a time, is denoted by the symbol P(n,r) or ^nP_r

a) Prove that $^nP_r = n!/(n-r)!$

b) If P(5,n) = 2 x P(6, n-1), find n

c) If $^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3: 5$, find n

d) In how many ways can 8 pictures be hung from 6 nails on a wall?

e) Prove that 0! = 1

16. a) Prove that the number of permutations of n objects, of which p_1 are alike of one kind, p_2 are alike of second kind, p_3 are alike of third kind … p_r are alike of rth kind such that $p_1 + p_2 + p_3 + … + p_r = n$, is $n!/(p_1! \ p_2! \ p_3! \ p_4! \ … \ p_r!)$.

b) Use the formula above to find the number of permutations of the letters in the words i) GOOGLE ii) BANGALORE iii) DARJEELING iv) MICROSOFT and v) MATHEMATICS.

b) For 'MATHEMATICS', in how many permutations are the vowels together?

17. In permutations, the order of arrangement of objects matters. If there are three letters xyz, the permutations of these letters taken two at a time are xy, yx, xz, zx, yz, zy, a total of 6 permutations. In contrast, in combinations, the order is immaterial. So, for the same example, the combinations of letters are xy, yz and zx, a total of 3 combinations. (yx is not a distinct combination because it is the same as xy, and so on.)

The number of combinations of n distinct objects, taken r at a time, is given by C(n,r) or $^nC_r = n! / (n-r)! \ r!$ (See if you can prove it. By symmetry, $^nC_r = {}^nC_{n-r}$. Note also that $^nC_r = {}^nP_r/r!$)

a) Evaluate $^{110}C_{106}$

b) If $^nP_r = 720$ and $^nC_r = 120$, find r

c) Prove that $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

18. a) You and your friends – a total of 20 - are having a birthday party. If each shakes hands with the other, how many handshakes occur in the party?

b) 25 players show up for playing cricket for Kolkata (formerly Calcutta) University. There are 11 batsmen (hitters), 7 bowlers (pitchers), 2 wicket-keepers (catchers) and 5 all-rounders (good in both batting and bowling). The coach decides that his 11 players must include 5 batsmen, 3 bowlers, 1 wicketkeeper and 2 all-rounders. In how many ways can the coach select his team?

19. a) Let P (6,4) and Q (2,12) be two given points. Find the slope of a line perpendicular to PQ.

    b) Find the equation of a line which is parallel to the y-axis and passes through (-4,3)

    c) Find the equation of a line that has y-intercept 4 and is perpendicular to the line joining (2,-3) and (4,2)

    d) Find the equation of the line joining the points (-1,3) and (4,-2)

20. a) A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1,5). Find its equation.

    b) Find the equation of the line which passes through the point (3,4) and the sum of its intercepts on the axes is 14.

    c) Prove that the lines $3x + y - 14 = 0$, $x - 2y = 0$ and $3x - 8y + 4 = 0$ are concurrent, that is, the lines pass through a common point.

    d) If two opposite vertices of a square are (1,2) and (5,8), find the coordinates of its other vertices.

21. a) If x, y, z are the sides of a triangle such that $x^2 + y^2 + z^2 = xy + yz + zx$, prove that the triangle must be equilateral.

    b) Find Pythagorean triangles whose sides are Fibonacci numbers.

22. The equation of a circle can be expressed as $(x-a)^2 + (y-b)^2 = c^2$, where (a,b) is the center of the circle, (x,y) is a point on the circumference, and c is the radius of the circle.

    a) Find the equation of a circle whose center is (3,-5) and radius 6.

    b) If the equations of the two diameters of a circle are $x - y = 5$ and $2x + y = 4$ and the radius of the circle is 7, find the equation of the circle.

    c) Find the center and radius of the circle $x^2 + y^2 + 6x - 4y + 4 = 0$

    d) Find the equation of the circle which passes through the point of intersection of the lines $x + 3y = 0$ and $2x - 7y = 0$ and whose center is the point of intersection of the lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$

    e) Find the equation of a circle, the coordinates of the end points of whose diameter are (-1,2) and (4,-3)

23. Definition: A conic (or conic section) is the locus of a point P which moves such that its distance from a fixed point F always bears a constant ratio to its

distance from a fixed line, all being on the same plane. The fixed point is the conic's focus, the fixed line its directrix, and the constant ratio e its eccentricity. When $e < 1$, the conic is an ellipse. When $e = 1$, it's a parabola. When $e > 1$, it's a hyperbola and when $e = 1$, it's a circle.

a) Find the equation of a parabola whose focus is (0,0) and whose directrix is the straight line $3x - 4y + 2 = 0$

b) If a parabolic reflector is 40 cm in diameter and 10 cm deep, find the equation of the reflector and its focus.

c) Prove that the equation of an ellipse centered at the origin, with major and minor axes of lengths 2a and 2b respectively is given by $x^2/a^2 + y^2/b^2 = 1$, where a and b are positive, $a^2 > b^2$ and $b^2 = a^2(1 - e^2)$. x and y are the coordinates of any point on the ellipse.

d) For the ellipse $25x^2 + 16y^2 = 400$, find the endpoints of the major and the minor axes. Graph the ellipse.

e) Derive the standard equation of a hyperbola centered at the origin: $x^2/a^2 - y^2/b^2 = 1$, where $b^2 = a^2(e^2 -1)$. Same symbols apply as in the standard equation of an ellipse given in c.

f) For the hyperbola $9x^2 - 4y^2 = 36$, find the endpoints of the vertices. Graph the hyperbola. How is graphing a hyperbola different from graphing an ellipse? What are asymptotes?

g) Does the equation $x^2 - 2y^2 - 2x + 8y - 1 = 0$ represent a hyperbola? If so, find the coordinates of its center, length of the axes, eccentricity, coordinates of the foci and vertices and equations of its directrices.

h) How are the reflective properties of parabolas used in the design of searchlights, automobile headlights, microphones, telescopes, radar and satellite dishes? Why are elliptical and hyperbolic shapes important in astronomy? How are elliptical properties used in the construction of ceilings, bridges, tunnels and in medical procedures? In what ways are the properties of hyperbolas used in architecture and in navigation? Use the Internet to explore the various practical uses of conic sections.

24. Expand using the binomial theorem:

a) $(x - 2y)^5$

b) $(1 - 2x + 3x^2)^3$

25. a) Find the 10^{th} term in the binomial expansion of $(2x^2 + 1/x)^{12}$

b) Find the coefficient of x^{10} in the binomial expansion of $(2x^2 - 3/x)^{11}$, where $x \neq 0$.

c) The coefficients of three consecutive terms in the binomial expansion of $(1+x)^n$ are in the ratio of 1: 7: 42. Find n.

d) Prove that dividing (6^n – 5n) by 25 always leaves a remainder of 1.

26. a) If the first few terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, …, express the Fibonacci sequence using abstract notations of a_1, a_2, … a_n, …, with n specified.

    b) The nth term of a sequence is given by $a_n = 2n + 7$. Show that it is an arithmetic progression (A.P) and find its 11^{th} term. Is $a_n = 2n^2 + n + 1$ an A.P.? Why or why not?

    c) Of the sequence 4, 9, 14, 19, … which term has the value of 299? What general formula can you use to find the value of a particular term in an A.P.?

27. a) Find the sum of all odd integers between 2 and 100 divisible by 3. Derive a general formula for the sum of a given number of terms in an A.P.

    b) If the first term of an A.P. is 2 and the sum of the first five terms is equal to one-fourth the sum of the next five terms, find the sum of the first 50 terms.

    c) The ratio of the sum of n terms of two A.P's is (7n+1) : (4n+27). Find the ratio of their nth terms.

    d) If a^2, b^2, c^2 form an A.P., prove that a/(b+c), b/(c+a), c/(a+b) also form an A.P.

28. In a geometric progression (G.P.), a_{n+1}/a_n = constant for all $n \in N$.

    a) Which term of the G.P. 2, 1, 1/2, 1/4, … has the value 1/512? What is the general formula you can use to solve problems like these?

    b) The fourth term of a G.P. is 8. Find the product of its first 6 terms.

    c) The sum of three consecutive numbers in a G.P. is 38 and their product is 1728, find the numbers.

    d) The product of the first three terms of a G.P. is 1000. If 6 is added to the second term and 7 to the third term, the terms become an A.P. Find the G.P.

    e) Find the sum of the series 5 + 55 + 555 + … to n terms.

    f) How many terms of the G.P. 1 + 4 + 16 + 64 + … will make the sum 5461?

    g) Prove that the sum of an infinite G.P. with the first term a and common ratio r is a/(1-r), where $|r| < 1$. What happens if $r >= 1$? Use the result to find the sum of the G.P. -5/4, -5/16, -5/64, … to infinity?

29. a) Prove that 1 + 2 + 3 + … + n = n(n+1)/2

    b) Prove that $1^2 + 2^2 + 3^2 + … + n^2 = n(n+1)(2n+1)/6$

c) Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \{n(n+1)\}^2$

d) Find the sum of the series $1^2 + 3^2 + 5^2 + \ldots$ to n terms.

e) Find the sum of n terms of the series whose nth term is $2n^2 - 3n + 5$

30. Evaluate:

a) $\lim_{x \to 2} (x^3 - 3x^2 + 4)/(x^4 - 8x^2 + 16)$

b) $\lim_{x \to 4} (x^2 - x - 12)^{18}/(x^3 - 8x^2 + 16x)^9$

c) $\lim_{x \to 1} (2x - 3)(\sqrt{x} - 1)/(2x^2 + x - 3)$

d) $\lim_{x \to \infty} (5x - 6)/(4x^2 + 9)^{1/2}$

e) $\lim_{n \to \infty} (1 + 2 + 3 + \ldots + n)/n^2$

f) $\lim_{x \to 0} (1 - \cos 2x)/x$

g) $\lim_{x \to 0} (\sin 2x + \sin 3x)/(2x + \sin 3x)$

h) $\lim_{x \to \pi} (1 + \sec^3 x)/\tan^2 x$

31. Differentiate the following functions with respect to x:

a) $x^2 + \sin x + 1$

b) $(\sqrt{x} + 1/\sqrt{x})^2$

c) $(x^2 + 1/x^2)^3$

d) $e^x \sin x + x^n \cos x$

e) $x \sin x \log x$

f) $e^x/(1 + \sin x)$

g) $(x^2 + 1)/(x + 1)$

32. a) Ten students take an algebra test. Their scores respectively are 54, 65, 45, 52, 68, 58, 62, 60 50 and 76. Find the variance and the standard deviation of the test scores.

    b) The mean and standard deviation of 20 observations of customers waiting in line at a bank at various times of the day are found to be 10 and 2 respectively. Later, it turned out that the observation of 8 was incorrect. Calculate the correct mean and standard deviation if i) the incorrect observation is omitted ii) if the incorrect observation is replaced with the correct observation of 12.

33. a) Two dice are thrown together. Find the probability of getting i) a total of 5. ii) a total of *at least* 5.

    b) What is the probability that i) in a group of 2 students, both will have the same birth day? ii) in a group of 4 students, *at least* 2 will have the same birth day? Does your answer change if you consider leap-year (29^{th} February) birthdays?

    c) What must be the minimum number of people in a gathering to ensure (100% probability) that two of them share the same birthday? What must be the minimum number of people in the gathering for a *50% probability* that at least two of them (any two) share the same birthday?

    d) Moushumi's birthday is October 2, same as India's "Father of the Nation," M. K. Gandhi. Moushumi is participating in an "International Day of Non-Violence" parade at Raj Ghat, Gandhi's memorial in New Delhi. What must be the minimum number of people in the parade for a 50% probability that at least one other person shares Moushumi's birthday?

34. a) An integer is selected at random from numbers ranging from 1 to 50. What is the probability that the selected integer is a multiple of 2 or 3 or 10?

    b) In a school in Mysuru (formerly Mysore) in the state of Karnataka, 40% students in the 11^{th}-grade study mathematics and 30% study physics. 10% study both mathematics and physics. If a student is selected at random form the class, find the probability that the student is studying mathematics, physics or both.